- # Engineering Mathematics
  # Diploma 1st year

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D. CO-ORDINATE GEOMETRY IN THREE DIMENSIONS

Three-dimensional rectangular co-ordinate system

It is known that the position of point in a plane is determined with reference two mutual perpendicular lines, called axes. Similarly, the position of a point in space is determined by three mutually perpendicular lines or axes. To specify a point in three-dimensional space, we take a fixed-point O, called origin and three mutually perpendicular lines through origin, called the X-axis, the Y-axis and the Z-axis. These three axes along with the 'origin' taken together are called the Rectangular co-ordinate system in three dimensions. The three axes are named as OX' , YOY' and ZOZ'. The axes in pairs determine three mutually perpendicular planes, XOY, YOZ and ZOX, called the coordinate planes. These planes are known as XY , YZ and ZX plane, respectively. Thus x- axis is perpendicular to YZ- plane , y- axis is perpendicular to ZX- plane and z- axis is perpendicular to XY- plane. The co-ordinate planes divide the space into eight equal compartments called octants. The sign of a point lying on the octant are mentioned below.

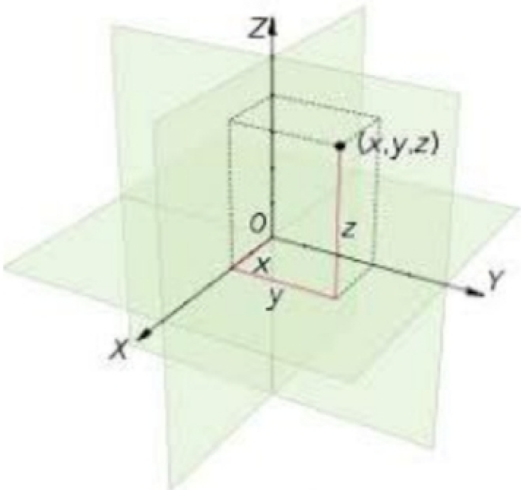


Fig 4.1

Octant	OXYZ	O$X'YZ$	OX$Y'Z$	OXYZ'	O$X'Y'Z$	O$X'YZ'$	OX$Y'Z'$	O$X'Y'Z'$
X	+	-	+	+	-	-	+	-
Y	+	+	-	+	-	+	-	-
Z	+	+	+	-	+	-	-	-

Let P be a point in space. Through P, three planes parallel to co- ordinate planes are drawn to cut the x-axis at A, y-axis at B ,and z-axis at C respectively. Let OA=x, OB=y and OC=z , these numbers x, y, z taken in order are called the co-ordinates of the point P in space and are denoted by P(x, y, z).

SPECIAL CASES

For any point P(x, y, z)

i) On XOY – plane , the z co-ordinate is equal to zero i.e P(x , y ,0)
ii) On YOZ – plane , the x co-ordinate is equal to zero i.e P(0 , y ,z)
iii) On ZOX – plane , the y co-ordinate is equal to zero i.e P(x , 0 ,z)
iv) On X- axis the value of y and z are zero (i.e y=z=0) i.e P(x , 0 ,0)
v) On Y- axis the value of x and z are zero (i.e x=z=0) i.e P(0 , y ,0)
vi) On Z- axis the value of x and y are zero (i.e x= y=0) i.e P(0 , 0 ,z)

Note :

The images of the point $P(x\,,y\,,z\,)$ with respect to XY , YZ and ZX plane are $(x\,,y\,,-z\,)$ $(-x\,,y\,,z\,)$ and $(x\,,-y\,,z\,)$ respectively.

Example: Find image of the point $(2\,,-3\,,-4)$ with respect to YZ – plane .

Sol:

the image of the point $(2\,,-3\,,-4)$ with respect to YZ – plane is $(-2\,,-3\,,-4)$

DISTANCE FORMULA

If $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are two given points, then distance between them is given by

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

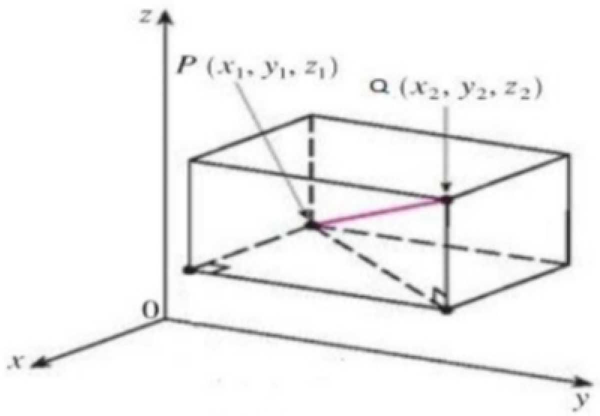


Fig 4.2

Example: Find the value of 'a' if distance between the points $P(0\,,2\,,0\,)$ and $Q(a\,,0\,,4\,)$ is 6.

Sol:

Given that $d=6,\ P(x_1,y_1,z_1)=(0\,,2\,,0\,)$ $Q(x_2,y_2,z_2)=(a\,,0\,,4\,)$

Using distance formula,

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$$d=\sqrt{(a-0)^2+(0-2)^2+(4-0)^2}$$

Or, $6=\sqrt{(a)^2+4+16}$

Or, $36=a^2+20$

Or, $a^2=16$

Or, $a = \pm 4$

Note: The perpendicular distance from a point $P(x, y, z)$ on positive direction

X – axis is $\sqrt{y^2 + z^2}$

Similarly the perpendicular distance from a point $P(x, y, z)$ on positive direction

Y – axis is $\sqrt{x^2 + z^2}$

The perpendicular distance from a point $P(x, y, z)$ on positive direction Z – axis

is $\sqrt{x^2 + y^2}$

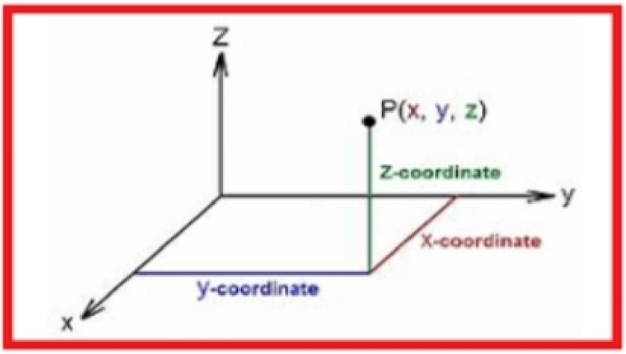


Fig 4.3

DIVISION FORMULA

Internal division formula

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two given points. Let $P(x, y, z)$ be a point which divides the line segment joining the points in the ratio m:n internally. Then co-ordinates of P are

$$x = \frac{mx_2 + nx_1}{m+n}, \quad y = \frac{my_2 + ny_1}{m+n}, \quad \text{and} \quad z = \frac{mz_2 + nz_1}{m+n}$$

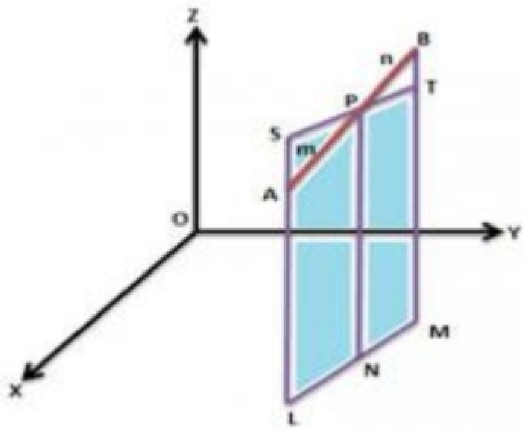


Fig 4.4

External division formula

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two given points. Let $P(x, y, z)$ be a point which divides the line segment in the ratio m:n externally. Then co-ordinates of P are

$$x = \frac{mx_2 - nx_1}{m-n}, \quad y = \frac{my_2 - ny_1}{m-n}, \quad \text{and} \quad z = \frac{mz_2 - nz_1}{m-n}$$

Midpoint formula

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two given points . Let $P(x\,, y\,, z)$ be the mid point of AB. Then the co-ordinates of P are

$$x = \frac{x_2+x_1}{2}, \quad y = \frac{y_2+y_1}{2}, \quad \text{and} \quad z = \frac{z_2+z_1}{2} \quad \text{(as m/n=1)}$$

DIRECTION COSINES AND DIRECTION RATIOS OF A LINE

Direction cosines:

The cosines of the angles made by a straight line with the positive directions of the co-ordinate axes are called the direction cosines of the line.

Let the line make an angle $\alpha\,, \beta\,\, and\, \gamma$ with positive direction of X ,Y and Z- axis respectively , then $\cos\alpha\,, \cos\beta\,, \cos\gamma$ are called direction cosines of that line. In short, it is written as $d.c^s$ Generally, $d.c^s$ of a line are denoted by $,m\,\, and\, n$.

i.e $l = \cos\alpha\,, m = \cos\beta \quad n = \cos\gamma$

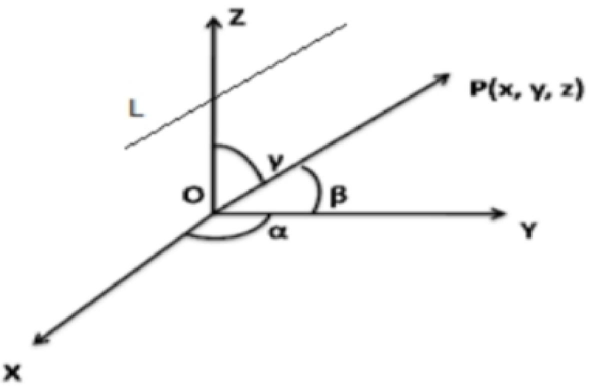


Fig 4.5

Notes :

i. The direction cosines of X – axis are $< 1\,, 0\,, 0 >$
   Since x- axis makes an angle 0^0 , 90^0 ,90^0 with co-ordinate axes.
ii. The direction cosines of X – axis in negative direction are $< -1\,, 0\,, 0 >$
   Since negative direction of x- axis makes an angle180^0, 90^0, 90^0 with co-ordinate axes.
iii. Similarly, the direction cosines of Y – axis are $< 0\,, 1\,, 0 >$
   and the direction cosines of Y – axis in negative direction are $< 0\,, -1\,, 0 >$
iv. The direction cosines of Z – axis are $< 0\,, 0\,, 1 >$
   and the direction cosines of Z – axis in negative direction are $< 0\,, 0\,, -1 >$

Direction ratios

Let a, b and c are three real numbers such that they are proportional to the direction cosine of a line.

I.e $\quad \frac{l}{a} = \frac{m}{b} = \frac{n}{c}$

Then these three numbers are called direction ratios of that line. The direction ratios of a line shortly written as $d.r^s$.

Notes:

i. Direction cosines of two parallel lines are equal but their direction ratios are proportional.

ii. Direction cosines of a line are unique (in respect of magnitude) where as a line may have infinite number of direction ratios .

The sum of squares of the direction cosines of any line is equal to unity

If *l, m,* and *n* are the direction cosines of a line, Then

$$l^2 + m^2 + n^2 = 1$$

Consider a line L having $d.c^s\ l, m\ and\ n$ and OP be a line segment of length r drawn from origin and parallel to L. Draw perpendicular PA , PB and PC from P on OX , OY and OZ respectively. If co-ordinates of P are (x , y, z)

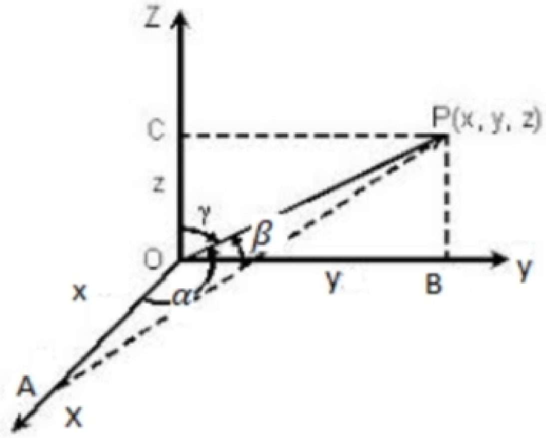


Fig 4.6

Then $\cos\alpha = \frac{x}{r}$ $\quad\cos\beta = \frac{y}{r}$ $\quad\cos\gamma = \frac{z}{r}$

i.e $\quad l = \frac{x}{r},\qquad m = \frac{y}{r},\qquad n = \frac{z}{r}$

i.e $\quad x = lr\qquad y = mr\qquad z = nr$

Now, $\quad l^2 + m^2 + n^2$

$$= \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2$$

$$= \frac{x^2+y^2+z^2}{r^2}$$

$$= \frac{r^2}{r^2} = 1 \qquad \text{(as } x^2+y^2+z^2 = r^2\text{)}$$

Therefore, $\quad l^2 + m^2 + n^2 = 1$

Direction cosines in terms of direction ratios

We know that $\quad \frac{l}{a} = \frac{m}{b} = \frac{n}{c}$

i.e $$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\pm\sqrt{a^2+b^2+c^2}}$$

i.e $$l = \frac{a}{\pm\sqrt{a^2+b^2+c^2}},\qquad m = \frac{b}{\pm\sqrt{a^2+b^2+c^2}},\qquad n = \frac{c}{\pm\sqrt{a^2+b^2+c^2}}$$

The two signs correspond to the two angles made by the line with co-ordinate axes.

Example : Find the direction cosines of a line whose direction ratios are 1 ,2,3

Sol:

Given that $a = 1\ , b = 2\ ,\ \ c = 3$

We know that

$$l = \frac{a}{\pm\sqrt{a^2+b^2+c^2}}\ ,\ m = \frac{b}{\pm\sqrt{a^2+b^2+c^2}}\ ,\qquad n = \frac{c}{\pm\sqrt{a^2+b^2+c^2}}$$

i.e $\quad l = \frac{1}{\pm\sqrt{1^2+2^2+3^2}} \qquad m = \frac{2}{\pm\sqrt{1^2+2^2+3^2}} \qquad n = \frac{3}{\pm\sqrt{1^2+2^2+3^2}}$

Or $\quad l = \frac{1}{\pm\sqrt{14}}, \qquad m = \frac{1}{\pm\sqrt{14}} \qquad n = \frac{1}{\pm\sqrt{14}}$

Hence, the direction ratios are $< \frac{1}{\pm\sqrt{14}}, \frac{2}{\pm\sqrt{14}}, \frac{3}{\pm\sqrt{14}} >$

Direction ratios when two points are given.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two given points. Then $d.r.s.$of that line segment PQ are $< x_2 - x_1\ , y_2 - y_1\ , z_2 - z_1\ >$

Projection of a line segment on another line

i. Projection when angle between the lines is given

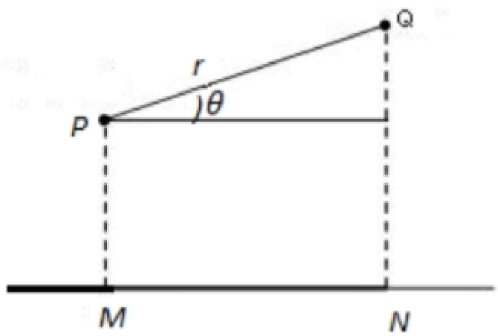


Fig 4.7

Let PQ be the line segment having length r and that makes an angle θ with another line MN , then projection of PQ on MN is

$$MN = PQ\ \cos\theta\ = r\ \cos\theta$$

ii. Projection of a line segment joining two given points

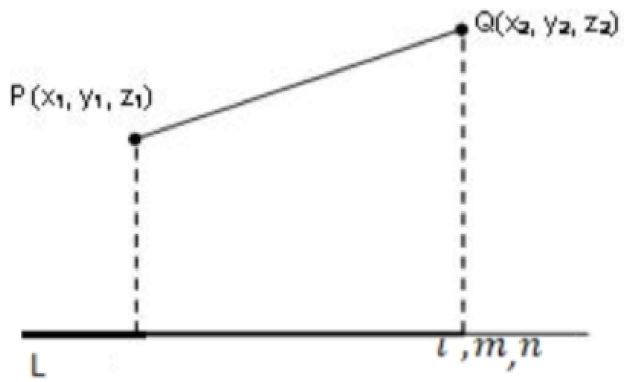


Fig 4.8

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two given points. Then projection of line segment PQ on the line L with direction cosines l , m ,n is

$$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

Angle between two lines

Let L_1 and L_2 be two lines having direction cosines are $< l_1, m_1, n_1 >$, $< l_2, m_2, n_2 >$ respectvilly. Let us draw OP and OQ parallel to L_1 and L_2. If θ is the angle between the lines, then

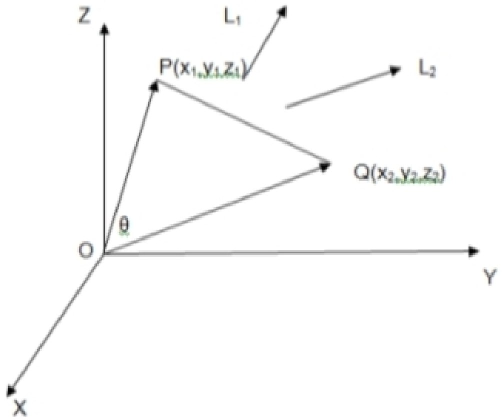


Fig 4.9

In Δ OPQ, $\cos\theta = \frac{OP^2+OQ^2-PQ^2}{2.OP.OQ}$

$$= \frac{(x_1^2+y_1^2+z_1^2)+(x_2^2+y_2^2+z_2^2)-\{(x_2-x_1)^2+(y_2-y_1)^2(z_2-z_1)^2\}}{2.OP.OQ}$$

$$= \frac{x_1x_2+y_1y_2+z_1z_2}{OP.OQ} = \frac{x_1}{OP}\frac{x_2}{OQ} + \frac{y_1}{OP}\frac{y_2}{OQ} + \frac{z_1}{OP}\frac{z_2}{OQ}$$

$= l_1l_2 + m_1m_2 + n_1n_2$ (by property of direction cosines)

$$= \frac{a_1a_2+b_1b_2+c_1c_2}{\pm\sqrt{a_1{}^2+b_1{}^2+c_1{}^2}\sqrt{a_2{}^2+b_2{}^2+c_2{}^2}}$$

Or , $\theta = \cos^{-1}(l_1l_2 + m_1m_2 + n_1n_2)$,

$$\theta = \cos^{-1}\left\{\frac{a_1a_2+b_1b_2+c_1c_2}{\pm\sqrt{a_1{}^2+b_1{}^2+c_1{}^2}\sqrt{a_2{}^2+b_2{}^2+c_2{}^2}}\right\}$$

Where $< a_1, b_1, c_1 >$ and $< a_2, b_2, c_2 >$ are direction ratios of L_1 and L_2 respectively.

Conditions of parallelism and perpendicularity:

i. **Perpendicular lines** :

If L_1 is perpendicular to L_2, then $\cos\theta=0$

Therefore,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

Or, $a_1a_2 + b_1b_2 + c_1c_2 = 0$

ii. Parallel lines

Since parallel lines have same direction cosines, it follows from definition of direction ratios that L_1 and L_2 are parallel if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

PLANE

Definition:

A plane is a surface such that the line joining any two points on the surface lies entirely on that surface.

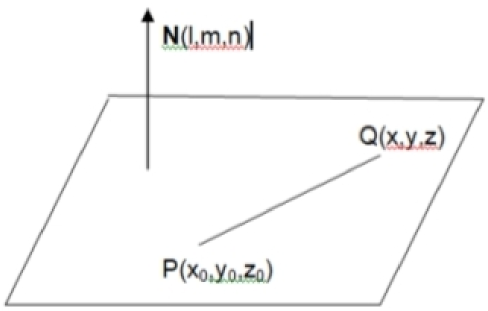


Fig 4.10

Equation of plane passing through a point and whose normal has given directional cosines.

Consider a point P(x_0, y_0, z_0) on the plane. Let (l, m, n) be the direction cosines of the normal to the plane. The direction ratios of the line joining the point P(x_0, y_0, z_0) and any point Q(x,y,z) on the surface are given as $(x-x_0)$, $(y-y_0)$, $(z-z_0)$ .Since the direction cosines of the normal to the plane are $l, m, n.$, by condition of perpendicularity

We have

$$l(x - x_0) + m(y - y_0) + n(z - z_0) = 0$$

Which is the equation of plane passing through point (x_0, y_0, z_0) and whose normal has direction cosines $l, m, n.$

Note:

In terms of direction ratios of the normal to the plane, the equation of plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Where a, b, c are direction ratios of the normal to the plane.

Example: Find the equation of plane passing through the point (2,1,3) & normal have direction ratio (1, 1, 1).

Sol:

Given that point (x_0, y_0, z_0) = (2, 1 , 3) and (a, b, c) are (1, 1 , 1)

Then equation plane is

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\Rightarrow 1(x-2) + 1(y-1) + 1(z-3) = 0$$

$$\Rightarrow x - 2 + y - 1 + z - 3 = 0$$

$$\Rightarrow x + y + z - 6 = 0$$

So, x – y + z = 6 is the required equation of plane.

General form of equation of plane

We have seen the equation of plane passing through point (x_0, y_0, z_0) and whose normal has direction cosines $l, m, n.$ is

$$l(x-x_0)+m(y-y_0)+n(z-z_0)=0$$

Or $\quad lx+my+nz-(lx_0+my_0+nz_0)=0$

Which is a 1^{st}. degree equation in x, y, z.

We can represent a plane by an equation of 1^{st}. degree in x, y and z.

Therefore, the equation of plane in general form is

$$Ax + By + Cz + D = 0,$$

If the plane passes through the origin, the equation becomes

$$Ax + By + Cz = 0,$$

Let a plane be given by

$$Ax + By + Cz + D = 0$$

If $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are two points on the plane, then

$$Ax_1 + By_1 + Cz_1 + D = 0$$

$$Ax_2 + By_2 + Cz_2 + D = 0$$

Subtraction of the above equations yields

$$A(x_1-x_2) + B(y_1-y_2) + C(z_1-z_2)=0$$

This shows the line with direction ratios (A,B,C) is perpendicular to the line with direction ratios $(x_1-x_2,\ y_1-y_2,\ z_1-z_2)$ i.e to PQ. But P and Q being any points on the plane, we see that the line with d.rs (A,B,C) is perpendicular to every line lying on the plane and hence is normal to the plane.

Thus the direction ratios of the normal to the plane, $Ax + By + Cz + D=0$ can be taken as (A,B,C) i.e coefficients of x, y, z respectively.

Equation of plane under different conditions

I. Equation of plane passing through three given points

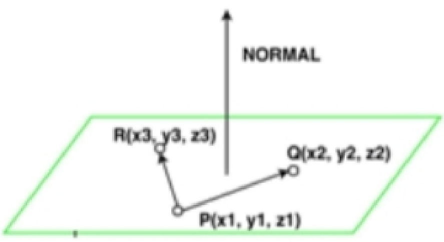


Fig 4.11

Let the plane pass through the points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Let the equation of the plane be

$$Ax + By + Cz + D = 0$$

Since it passes through the points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

We have

$$Ax_1 + By_1 + Cz_1 + D = 0$$
$$Ax_2 + By_2 + Cz_2 + D = 0$$
$$Ax_3 + By_3 + Cz_3 + D = 0$$

Subtraction of equations yield

$$A(x - x_1) + B(y - y_1) + C(z - z_{1)} = 0$$
$$A(x - x_2) + B(y - y_2) + C(z - z_2) = 0$$
$$A(x - x_3) + B(y - y_3) + C(z - z_3) = 0$$

Eliminating the constants *A, B,C,*
We get

$$\begin{vmatrix} x - x_1 & y - y_{1,} & z - z_1 \\ x_{2,} - x_{1,} & y_{2,} - y_{1,} & z_2 - z_1 \\ x_{3,} - x_{1,} & y_{3,} - y_{1,} & z_3 - z_1 \end{vmatrix} = 0$$

Which is the equation of the plane.

Note: co- planar condition for four points

Four given points $(x_{1,}y_{1,}z_1)$, $(x_{2,}y_{2,}z_2)$, $(x_{3,}y_{3,}z_3)$ & $(x_{4,}y_{4,}z_4)$, are co-planar, if

$$\begin{vmatrix} x_4 - x_1 & y_4 - y_{1,} & z_4 - z_1 \\ x_{2,} - x_{1,} & y_{2,} - y_{1,} & z_2 - z_1 \\ x_{3,} - x_{1,} & y_{3,} - y_{1,} & z_3 - z_1 \end{vmatrix} = 0$$

Example : Find equation of plane passing through the points(1,3,1),(2,2,1) & (3,2,4).

Sol: We know that equation of plane passing through three given points is

$$\begin{vmatrix} x - x_1 & y - y_{1,} & z - z_1 \\ x_{2,} - x_{1,} & y_{2,} - y_{1,} & z_2 - z_1 \\ x_{3,} - x_{1,} & y_{3,} - y_{1,} & z_3 - z_1 \end{vmatrix} = 0$$

Or,
$$\begin{vmatrix} x - 1 & y - 3 & z - 1 \\ 2 - 1 & 2 - 3 & 1 - 1 \\ 3 - 1 & 2 - 3 & 4 - 1 \end{vmatrix} = 0$$

Or,
$$\begin{vmatrix} x - 1 & y - 3 & z - 1 \\ 1 & -1 & 0 \\ 2 & -1 & 3 \end{vmatrix} = 0$$

Or
$$(x - 1)(-3) - (y - 3)(3) + (z - 1)(-1 + 2) = 0$$

Or
$$3x + 3y - z - 11 = 0$$

Which is the required equation of the plane.

II. Equation of a plane parallel to a give plane

Let Ax + By + Cz + D = 0 be the equation of a plane. Then the equation any plane parallel to the above plane is

Ax + By + Cz + K = 0, Where K is constant.

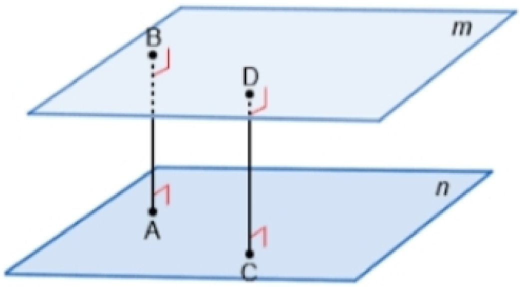


Fig 4.12

Note: Since parallel planes have the same normal, in the equation of parallel plane the coefficients of x, y ,z do not change. Only the constant D will be different.

Example : Find the equation of plane passing through the point (2 , -2 , -1) and parallel to the plane 2x + y - 3z - 2 = 0.

Sol: We know that equation any plane parallel to the plane 2x + y - 3z - 2 = 0 is

2x + y - 3z + K = 0 (1)

Since the plane passes through the point (2,-2,-1) it will satisfy the equation (1),

Thus we have,

$2 \times 2 - 2 - 3 \times (-1) + K = 0$

Or, K + 5 =0

Or, K = -5

Putting the above value in eqn(1)

We get

2x + y – 3 z - 5 =0

Which represents the required plane

III. Equation of plane passing through intersection of two given planes

Let $A_1X + B_1Y + C_1Z + D_1 = 0$ and $A_2X + B_2Y + C_2Z + D_2 = 0$ be two given planes π_1 and π_2 . Then equation of plane passing through intersection of above two plane is given by

$$A_1X + B_1Y + C_1Z + D_1 + \mathrm{K}(A_2X + B_2Y + C_2Z + D_2) = 0$$

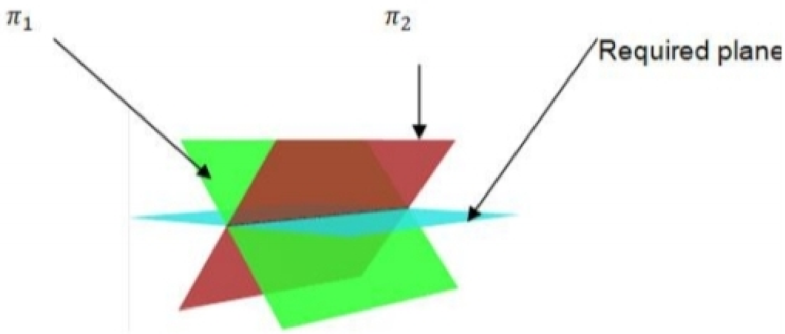


Fig 4.13

Example: Find the equation of plane passing through intersection of planes $2x + 3y - 4z + 1 = 0$, $3x - y + z + 2 = 0$ and passing through the point (3 , 2 , 1).

Sol: We known that equation any plane passing through intersection of the two planes $2x + 3y - 4z + 1 = 0$ and $3x - y + z + 2 = 0$ is

$(2x + 3y - 4z + 1)$ +K $(3x - y + z + 2) = 0$ (1)

Since the plane (1) passes through the point (3 , 2 , 1). , we get

$(2.3 + 3.2 - 4.1 + 1) + K\,(3.3 - 2 + 1 + 2) = 0$

i.e $9 + 10K = 0$

Or, $10K = -9$

Or $K = -9/10$

Putting the value of K in eqn (1), we get the equation of the required plane

$(2x + 3y - 4z + 1) - 9/10(3x - y + z + 2) = 0$

i.e $7x - 39y + 49z + 8 = 0$

Equations of Plane in different forms

I. Equation of plane in intercept form

Let a plane intercept the coordinate axes, X , Y & Z , at a distance a, b and c from origin respectively.

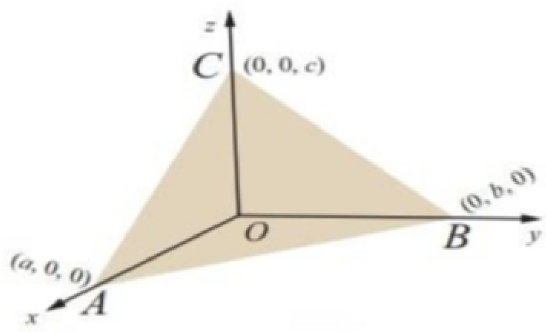


Fig 4.14

Then, the plane passes through the points A(a,0,0),B(0,b,0) and C(0,0,c)

Hence, the equation of the plane is given by

$$\begin{vmatrix} x-a & y-0 & z-0 \\ 0-a & b-0 & 0-0 \\ 0-a & 0-0 & c-0 \end{vmatrix} = 0$$

This on simplification gives

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Which is the equation of plane in **intercept form.**

Example: Find the equation of plane , whose X ,Y ,Z axis intercepts are 1,2,3 respectively.

Sol : Given a = 1, b = 2 & c = 3

So the equation of plane is $\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$

Or, $6x + 3y + 2z - 6 = 0$

II. Equation of plane in Normal form

Let p be the length of the perpendicular ON from the origin on the plane and let *l* , *m, n* be its direction cosines. Then the coordinates of the foot of the perpendicular N, are (*lp, mp, np*)

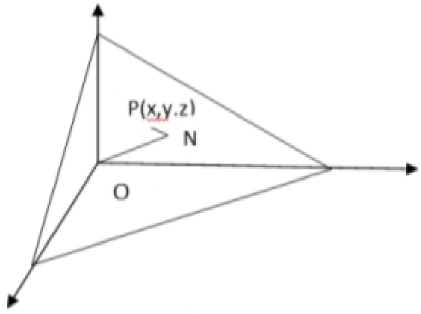


Fig 4.15

If P(x, y, z) be any point on the plane, then the direction cosines of NP are (x-lp, y-np, z-np)
Since ON is perpendicular to the plane, it is also perpendicular to NP
Hence,

$$l(x - lp) + m(y - np) + n(z - np) = 0$$

Or $$lx + my + nz = (l^2 + m^2 + n^2)p$$

Or, $$lx + my + nz = p$$

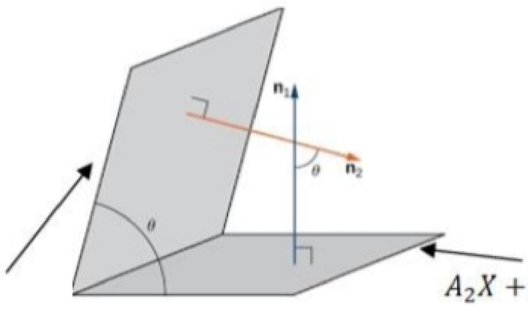
Is the equation of plane in **Normal form.**

Angle between two intersecting planes

Let us consider two intersecting planes

$$A_1X + B_1Y + C_1Z + D_1 = 0$$
$$A_2X + B_2Y + C_2Z + D_2 = 0$$

$A_1X + B_1Y + C_1Z + D_1 = 0$



$A_2X + B_2Y + C_2Z + D_2 = 0$

Fig 4.16

Let θ be the angle between the planes. It is obvious that θ also measures an angle between the normals to the planes.
Therefore,

$$\theta = \cos^{-1}\left(\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{{A_1}^2 + {B_1}^2 + {C_1}^2}\sqrt{{A_2}^2 + {B_2}^2 + {C_2}^2}}\right)$$

Note: Two planes $A_1X + B_1Y + C_1Z + D_1 = 0$, $A_2X + B_2Y + C_2Z + D_2 = 0$

I. Are **parallel** to each other if their normals are parallel

i.e $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

(This ratio can be conveniently taken as 1 while solving problems)

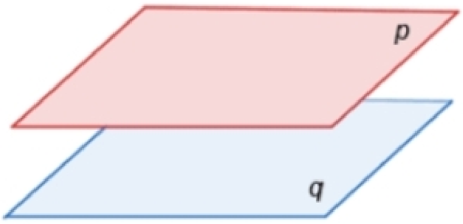


Fig 4.17

II. Are **perpendicular** to each other if their normals are perpendicular to each other

i.e $A_1A_2 + B_1B_2 + C_1C_2 = 0$

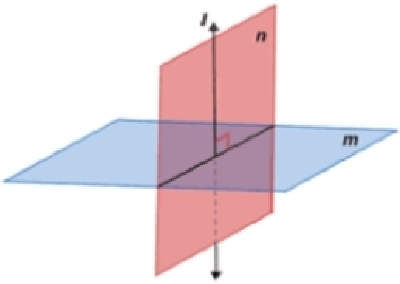
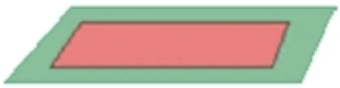


Fig 4.18

III. Are **coincident**

If $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}$



Coincident

Fig 4.19

Perpendicular distance of a point from the plane

Let P(x_0, y_0, z_0) be a point on the plane, Ax + By + Cz + D = 0. Then perpendicular distance from the point to the plane is

$$p = \frac{Ax_0 + By_0 + Cz_0 + D}{\pm\sqrt{A^2 + B^2 + C^2}}$$

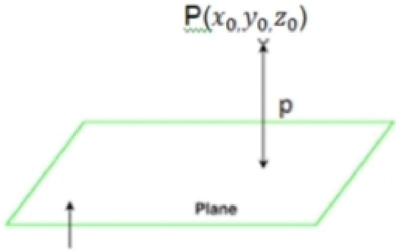


Fig 4.20

Note : The perpendicular distance from origin (0, 0, 0) to the plane Ax + By + Cz + D = 0 is,

$$p = \frac{D}{\pm\sqrt{A^2 + B^2 + C^2}}$$

EXERCISE

1. 02 Mark Questions

i. Find the distance of the point P(1,2,3) from z axis.
ii. Find the direction cosines of the line joining the points (8 ,-1 ,5) and (2 , -4 ,3).
iii. Determine the direction cosines of the line equally inclined to both the axes.
iv. Find the number of lines making equal angles with coordinate axes.
v. If a line a line is perpendicular to z-axis and makes an angle measuring 60^0 with x –axis then find the angle it makes with y-axis.
vi. Find the projection of line segment joining (1,3,-1) and (3,2,4) on z axis.
vii. Find the image of the point (2,-4,7) with respect to xz plane.
viii. For what value of z, the distance between the points (-1,1,2) and (-1,-1,z) is 4.
ix. Find the centre of the sphere $x^2+y^2+(z+2)^2 = 0$.
x. If the centre and radius of a sphere are (1,0,0) and 2 respectively, then find the equation of the sphere.
xi. If the segment of line joining the points (1,0,0) and (0,0,1) is a diameter of a sphere, then find equation of the sphere.

2. 05 Mark Questions

i. Prove that angle between two main diagonals of cube is $\cos^{-1}\frac{1}{3}$
ii. Find the ratio in which the line through (1 ,-1 ,3) and (2 , -4 ,1) is divided by XY- plane.
iii. Find the ratio in which the line through (1 ,-1 ,3) and (2 , -4 ,1) is divided by YZ- plane.
iv. If P(x , y , 2) lies on the line through (1 ,-1 ,0) and (2 , 1 ,1). Find the values of x and y.
v. Find the ratio in which the line joining the points (2 , -3 , 1) and (3 , -4 , -5) is divided by the locus $2x - y + 3z - 4 = 0$.
vi. Find the foot of perpendicular drawn from the point (1 ,1 , 2) on the line joining (1 ,4 ,2) and (2 , 3 ,1).
vii. Find the value of k, if the distance between the points (-1 ,-1 ,k) and (1 ,-1 ,1) is 2.
viii. Find the value of 'a' such that two planes $2x + y + az - 2 = 0$ and $3x - y + 5z - 2 = 0$ are perpendicular to each other.
ix. Find angle between the planes $3x - y + 5z - 2 = 0$ and $3x - y + 5z - 2 = 0$
x. Find the equation of a plane passing through the points (1,2,3), (1,-2,-3) and perpendicular to the plane $3x - 3y + 5z - 2 = 0$.
xi. Find the equation of plane passing through intersection of planes $3x + y + z - 2 = 0$ and $x - 2y + 3z - 1 = 0$ and parallel to the plane $x - y + z - 6 = 0$.
xii. Find the equation of plane passing through intersection of planes $3x + 2y + z + 2 = 0$ and $x - 2y + 2z - 3 = 0$ and perpendicular to the plane $4x - y + 3z - 7 = 0$.